ENEE 351 — Jeanne Perdón

Lecture 12

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RANDOMIZED_QUICKSORT (A, p, r)
    If p < r
        q = RANDOMIZED_PARTITION (A, p, r);
        RANDOMIZED_QUICKSORT (A, p, q-1);
        RANDOMIZED_QUICKSORT (A, q+1, r);
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• (q = RANDOMIZED...) → Define $X_q = \begin{cases} 1, & \text{if the } q^{th} \text{ smallest element is picked as a pivot} \\ 0, & \text{otherwise} \end{cases}$

21	10	1	5	15	0	17	25

↑ (under 5)

$x_1 = 0$ 0 1 5 . . .

$x_2 = 0$ ↓ x_3

$x_3 = 1$

$x_4 = 0$

$x_5 = 0$

x_6 .

x_7 .

x_8 .

elements on left

$$T(n) = X_1\big(T(0) + T(n-1)\big) + X_2\big(T(1) + T(n-2)\big) + X_3\big(T(2) + T(n-3)\big) + \cdots X_n\big(T(n-1) + T(0)\big) + n$$

$$= \cancel{T(0)}^{\,0}[X_1 + X_n] + T(1)[X_2 + X_{n-1}] + \cdots + T(n-1)[X_1 + X_n] + n$$

$$\boxed{T(n) = \sum_{q=1}^{n-1} T(q)\,[X_{q+1} + X_{n-q}] + n}$$

☆ $\boxed{\begin{array}{l} X : x_1, x_2 \cdots x_n \\ \quad P_1, P_2 \cdots P_n \\ \mathbb{E}(x) = \sum_{i=1}^{n} X_i P_i \end{array}}$

Expectation:

$$\mathbb{E}(X_q) = \frac{1}{n}$$

$$\mathbb{E}(X_q) = \frac{1}{n}\cdot 1 + 0 \cdots$$

Example:

$$X = \begin{cases} a, & P \\ b, & q \end{cases} \implies \mathbb{E}(X) = ap + bq$$

$$\mathbb{E}(T(n)) = \mathbb{E}\left(\sum_{q=1}^{n-1} T(q)\,[X_{q+1} + X_{n-q}]\right) + n$$

$$= \sum_{q=1}^{n-1} \mathbb{E}\big(T(q)(X_{q+1} + X_{n-q})\big) = \sum_{q=1}^{n} \mathbb{E}\big(T(q)\,X_{q+1} + T(q)\,X_{n-q}\big)$$

$$= \sum_{q=1}^{n-1} \left(\mathbb{E}(\underbrace{T(q) X_{q+1}}_{\frac{\mathbb{E}(T(q))}{n}}) + \mathbb{E}(T(q) X_{n-q}) \right)$$

$$= \frac{2}{n} \sum_{q=1}^{n-1} T(q) + n$$

$$\mathbb{E}(T(q) X_q) = \frac{\mathbb{E}(T(q))}{n}$$

$$\mathbb{E}(T(q) X_{n-q}) = \frac{T(q)}{n}$$

$$\mathbb{E}(T(n)) = \frac{2}{n} \sum_{q=1}^{n-1} \mathbb{E}(T(q)) + n$$

$$\boxed{\mathbb{E}(T(n)) = \begin{cases} 1, & \text{if } n=1 \\ \frac{2}{n} \sum_{q=1}^{n-1} \mathbb{E}(T(q)) + n, & n>1 \end{cases}}$$

- We need to prove that this solves to $O(n \log n)$
  $\exists c : \mathbb{E}(T(n)) \leq cn \log n + 1$

  – *Base case*: ✓ (plug $n=1$)

  – *Induction step*:
  Assume $\mathbb{E}[T(q)] \leq c\, q \log q + 1$, for all $q < n$

$$\mathbb{E}(T(n)) = \frac{2}{n} \sum_{q=1}^{n-1} \mathbb{E}(T(q)) + n \leq \frac{2}{n} \sum_{q=1}^{n-1} (c \cdot q \log q + 1) + n$$

$$= \frac{2c}{n} \sum_{q=1}^{n-1} q \log q + 2n - 1$$

Bound: $\sum_{q=1}^{n-1} q \log q \leq \int_1^n x \log x \, dx$

$$= \frac{1}{\ln 2} \int_1^n x \ln x \, dx = \frac{1}{\ln 2} \int_1^n \left(\frac{x^2}{2}\right)' \ln x \, dx$$

Integration by parts:
$\int f'(x) g(x) dx$
$= f(x) g(x)$
$- \int f(x) g'(x) dx$

$$= \frac{1}{\ln 2} \left[\frac{x^2}{2} \ln x - \int_1^n \frac{x^{\cancel{2}}}{2} \left(\frac{1}{\cancel{x}}\right) dx \right]$$

$$= \frac{1}{\ln 2} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right] = \frac{1}{\ln 2} \left(\frac{n^2}{2} \ln n - \frac{n^2}{4} + \frac{1}{4} \right)$$

$$= \boxed{\frac{n^2}{2} \log n - \frac{n^2}{4 \ln 2} + \frac{1}{4 \ln 2}}$$

$$\mathbb{E}(T(n) = \frac{2}{n}\sum_{q=1}^{n-1} \mathbb{E}(T(q)) + n \leq \frac{2}{n}\sum_{q=1}^{n-1}\left(c \cdot q \log q + 1\right) + n$$

$$= \frac{2c}{n}\sum_{q=1}^{n-1} q \log q + 2n - 1$$

$$\leq \frac{\cancel{2}c}{\cancel{n}}\left(\frac{n^{\cancel{2}}}{\cancel{2}}\log n - \frac{n^{\cancel{2}}}{\cancelto{2}{4}\ln 2} + \frac{1}{\cancelto{2}{4}\ln 2}\right) + 2n - 1$$

let $c = 100$

$$= c \cdot n \log n - \frac{cn}{2\ln 2} + \cancelto{n \to \infty}{\frac{c}{2n\ln 2}} + 2n - 1$$

$$\leq c\, n \log n$$

$$\leq cn \lg n + 1$$