can show that the maximum lo… $1/n$ for sufficiently large n via a direct calculation and … loose bound; although the maximum load is in fact $\Omega(\ln n/\ln\ln n)$ with probability close to 1 (as we show later), the constant factor 3 we use here is chosen to simplify the argument and could be reduced with more care.

Lemma 5.1: *When n balls are thrown independently and uniformly at random into n bins, the probability that the maximum load is more than $3\ln n/\ln\ln n$ is at most $1/n$ for n sufficiently large.*

Proof: The probability that bin 1 receives at least M balls is at most

$$\binom{n}{M}\left(\frac{1}{n}\right)^M.$$

This follows from a union bound; there are $\binom{n}{M}$ distinct sets of M balls, and for any se[t] of M balls the probability that all land in bin 1 is $(1/n)^M$. We now use the inequalities

$$\binom{n}{M}\left(\frac{1}{n}\right)^M \le \frac{1}{M!} \le \left(\frac{\mathrm{e}}{M}\right)^M.$$

Here the second inequality is a consequence of the following general bound on facto-rials: since

$$\frac{k^k}{k!} < \sum_{i=0}^{\infty}\frac{k^i}{i!} = \mathrm{e}^k,$$

we have

$$k! > \left(\frac{k}{\mathrm{e}}\right)^k.$$

Applying a union bound again allows us to find that, for $M \ge 3\ln n/\ln\ln n$, the pr[ob]ability that any bin receives at least M balls is bounded above by

$$\begin{aligned}
n\left(\frac{\mathrm{e}}{M}\right)^M &\le n\left(\frac{\mathrm{e}\ln\ln n}{3\ln n}\right)^{3\ln n/\ln\ln n}\\
&\le n\left(\frac{\ln\ln n}{\ln n}\right)^{3\ln n/\ln\ln n}\\
&= \mathrm{e}^{\ln n}\left(\mathrm{e}^{\ln\ln\ln n-\ln\ln n}\right)^{3\ln n/\ln\ln n}\\
&= \mathrm{e}^{-2\ln n+3(\ln n)(\ln\ln\ln n)/\ln\ln n}\\
&\le \frac{1}{n}
\end{aligned}$$

for n sufficiently large.